Can codimension-two branes solve the cosmological constant problem?

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It has been suggested that codimension-two braneworlds might naturally explain the vanishing of the 4D effective cosmological constant, due to the automatic relation between the deficit angle and the brane tension. To investigate whether this cancellation happens dynamically, and within the context of a realistic cosmology, we study a codimension-two braneworld with spherical extra dimensions compactified by magnetic flux. Assuming Einstein gravity, we show that when the brane contains matter with an arbitrary equation of state, the 4D metric components are not regular at the brane, unless the brane has nonzero thickness. We construct explicit 6D solutions with thick branes, treating the brane matter as a perturbation, and find that the universe expands consistently with standard Friedmann-Robertson-Walker (FRW) cosmology. The relation between the brane tension and the bulk deficit angle becomes $\Delta = 2\pi G_6(\rho - 3p)$ for a general equation of state. However, this relation does not imply a self-tuning of the effective 4D cosmological constant to zero; perturbations of the brane tension in a static solution lead to deSitter or anti-deSitter braneworlds. Our results thus confirm other recent work showing that codimension-two braneworlds in nonsupersymmetric Einstein gravity do not lead to a dynamical relaxation of the cosmological constant, but they leave open the possibility that supersymmetric versions can be compatible with self-tuning.

I. INTRODUCTION

The impressive correspondence between the ever-increasing amount of high-precision cosmological data [1, 2] and the theoretical predictions of the Λ-Cold Dark Matter (ΛCDM) model is undeniably one of the greatest success stories of modern physics. Casting a shadow over this success however is the nagging fact that there is to date no satisfactory understanding of just what the dark energy (Λ) component is, and why its properties are what they are. Working under the assumption that this term is a cosmological constant due to vacuum energy, naive estimates of its size come out a staggering $60 - 120$ orders of magnitude too large (the number depending on one's beliefs about supersymmetry), which until recently led most physicists to suspect that there must exist some as-yet undiscovered mechanism which exactly cancels its effect. The fact that observations now tell us that this term is *not* exactly zero, but nevertheless extraordinarily small by particle physics standards, means that finding a theoretical framework where we can understand its scale and origin has become a pressing challenge for the theoretical physics community.

One promising line of attack on the cosmological constant problem, which has recently garnered renewed interest, centers around ideas which arose some twenty years ago with the work of Rubakov and Shaposhnikov [3]. These authors realized that in certain six-dimensional models, the expansion rate of the three large spatial dimensions is an integration constant and thus independent of the vacuum energy. Several papers followed, e.g. [4]-[7], which studied features such as stability and cosmology in these models.

Lately, the advent of braneworld scenarios [8]-[11] has rekindled interest in these ideas as possible ways of addressing the cosmological constant problem [12]-[17]. In the new perspective, the old six dimensional models can be recast as codimension-two braneworlds that have the interesting property that the brane tension, rather than leading to expansion, induces a deficit angle in the bulk. Thus the Hubble expansion measured by observers on the brane would be insensitive to the vacuum energy associated with any field theory confined to that brane. This has led to the hope that there might exist a mechanism whereby if the brane tension suddenly changed, as in a phase transition, the deficit angle would dynamically relax to a value which cancels the tension's contribution to the expansion rate, thus providing at least a partial solution to the cosmological constant problem. (It would be only a partial solution since it is still necessary to do one fine-tuning of bulk contributions to the effective 4D cosmlogical constant.)

However there are open questions concerning this scenario. It is not at all clear that such a tuning mechanism is really possible within Einstein gravity [18]-[20]. Furthermore, it has been argued that it is impossible to find solutions with any matter sources on the brane except for those with the equation of state of pure tension [15, 21], which is a major obstruction to building more general models that would allow one to recover FRW cosmology at late times.

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Much effort has been devoted to studying the cosmology of codimension-one branes, and the associated corrections to standard cosmology are well-known [22]-[24]. Lately codimension-two braneworlds have received increasingly more attention, but most of this work has focused on static solutions or issues of stability [25]-[39], while the papers addressing cosmology have considered almost exclusively only pure tension branes (see, however, [21],[40]-[43] for recent proposals for generalizing the brane equation of state, and [44] for a different perspective on the matter).

In this paper, we argue that the apparent obstruction to finding solutions with a more general equation of state for codimension-two branes is not a problem in principle; rather it stems from an unrealistic expectation that the bulk geometry should be regular in the vicinity of the brane. This singularity happens to be absent for branes whose equation of state is that of pure tension, $\rho = -p$. But in general, sources of codimension p in gravity and electrostatics are well known to give singular solutions $\sim r^{2-p}$, so it should be no surprise to find a logarithmic singularity in the metric in the codimension-2 case. To study this in a controlled manner, we will regularize the singularity by giving the brane a finite thickness r_0, and eventually take the limit $r_0 \to 0$.

In terms of physics, our goal is to answer two specific questions:

- Can the deficit angle dynamically relax to a value which cancels the effect of the brane tension?

- Do these models lead to FRW cosmology when the matter on the brane has an arbitrary equation of state?

The plan of the paper is as follows: in Sec. II, we review the argument behind the claim that one can only have pure tension on codimension-two delta function branes in Einstein gravity. In Sec. III we introduce the thick-brane setup we will be using to circumvent this difficulty. In Sec. IV, we review the known static background solution which we will be perturbing to obtain cosmological solutions. We present the linearized perturbative equations of motion and the Friedmann equations which follow from their solutions in Section V. Section VI rephrases these results in terms of effective four-dimensional quantities that are relevant to observers living on the brane, and derives the Friedmann equations. The reader who is interested in the main results of the paper may wish to skip directly to this point. We discuss our results in Sec. VII and give conclusions in Sec. VIII. Full solutions to the perturbed equations of motion are quite lengthy, and are provided to the interested reader in the appendices.

It should be emphasized that supersymmetric versions of the model we are considering [16, 45, 46] may be more promising, and represent a worthwhile subject for further research.

II. THE TROUBLE WITH DELTA FUNCTION BRANES

It has already been pointed out in [15] and confirmed by [21] that in Einstein gravity, codimension-two delta function branes appear to be incompatible with the matter on the brane having a general equation of state. We repeat here the crux of the argument.

We start with the most general six dimensional metric with the two extra dimensions having axial symmetry

$$ds^2 = -N^2(\bar{r},\bar{t})d\bar{t}^2 + A^2(\bar{r},\bar{t})d\vec{x}^2 + B^2(\bar{r},\bar{t})d\bar{r}^2 + C^2(\bar{r},\bar{t})d\theta^2 + 2E(\bar{r},\bar{t})d\bar{r}d\bar{t} \tag{1}$$

where t and $\vec{x}$ correspond to the usual four large dimensions, while r and θ span the two dimensional internal space. This metric can be put into the following form through a suitable transformation of the coordinates $\bar{r}(r,t)$ and $\bar{t}(r,t)$:

$$-n^2(r,t)dt^2 + a^2(r,t)d\vec{x}^2 + f^2(r,t)(dr^2 + r^2d\theta^2). \tag{2}$$

The equations relevant to our discussion are the (tt) and (xx) components of the Einstein equations, (here and in the rest of the text, primes stand for derivatives with respect to r while dots represent derivatives with respect to t),

$$\frac{1}{f^2}\left[\nabla^2\ln(f) + 3\nabla^2\ln(a) + 6\left(\frac{a'}{a}\right)^2\right] - \frac{1}{n^2}\left[3\left(\frac{\dot{a}}{a}\right)^2 + \left(\frac{\dot{f}}{f}\right)^2 + 6\frac{\dot{a}}{a}\frac{\dot{f}}{f}\right] = 8\pi G_6 T^t_t \tag{3}$$

$$\left[\nabla^2\ln(f) + \nabla^2\ln(n) + 2\nabla^2\ln(a) + 2\frac{n'}{n}\frac{a'}{a} + 2\left(\frac{a'}{a}\right)^2 + \left(\frac{n'}{n}\right)^2\right]$$

$$+\frac{1}{n^2}\left[2\frac{\dot{a}}{a}\frac{\dot{n}}{n} - 2\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 - 4\frac{\dot{a}}{a}\frac{\dot{f}}{f} - \left(\frac{\dot{f}}{f}\right)^2 - 2\frac{\ddot{f}}{f} + 2\frac{\dot{n}}{n}\frac{\dot{f}}{f}\right] = 8\pi G_6 T^x_x \tag{4}$$

where ∇^2 is the two-dimensional Laplacian for the coordinates r and θ. Supposing there is a codimension-two delta function brane at the origin of the internal space, T_t^t and T_x^x contain the singular terms $-\rho(t)\delta(r)/(2\pi f^2(r,t)r)$ and $p(t)\delta(r)/(2\pi f^2(r,t)r)$ respectively. Since in two dimensions the delta function can be written as

$$\delta(r) = 2\pi r \nabla^2 \ln(r), \tag{5}$$

the singular parts of the Einstein tensor must come from the terms $\nabla^2 \ln(f)$, $\nabla^2 \ln(a)$ or $\nabla^2 \ln(n)$. It is not immediately obvious how to interpret having $a(0,t) \sim r^{f_a(t)}$ or $n(0,t) \sim r^{f_n(t)}$ if we are to put the standard model on a 3-brane at the origin, since the metric is either vanishing or singular (depending on the sign of the exponent) there. If we consider this to be a problem, the Einstein tensor only allows for

$$G_t^t|_{\text{sing}} = G_x^x|_{\text{sing}} \tag{6}$$

and we are led to conclude that codimension-two delta function branes can only have a stress energy tensor of the form $T_t^t = T_x^x$ (with $T_r^r = T_\theta^\theta = 0$). Furthermore, the deficit angle is related to an integral over the singular part of the Einstein tensor, and is therefore rigidly related to the brane tension through the Einstein equations. This prevents us from being able to meaningfully ask the question "what happens if there is a sudden phase transition on the brane, so that the tension and deficit angle become detuned?" unless we somehow regularize the behaviour of the metric at the brane.

One possible regularization is to give the brane a finite thickness. By smoothing the singularity in the source, we remove the necessity of making the metric components vanish or become singular at the position of the brane. This is the approach we will follow in this paper, since we feel that it is less contrived than certain modifications to general relativity which could alternatively be invoked, and it allows for a better physical understanding of the apparent obstruction to zero-thickness branes with a general equation of state.

Among the other possibilities that invoke modified gravity, which we will not pursue here, is the addition of an intrinsic curvature term to the brane action, as in Dvali-Gabadadze-Porrati braneworlds (see, e.g. [47, 48]). It has been argued that this scenario should arise naturally since the matter on the brane can induce a 4D kinetic term for gravitons through loop corrections [47]. Such terms can allow one to obtain additional delta function contributions in the equations of motion to match with a brane where $\rho \neq -p$. In this scenario, the deficit angle acts on the brane like a negative cosmological constant which, in contrast with what happens in the absence of the intrinsic curvature term, is no longer forced by the equations of motion to be equal to the brane tension.

A second possibility, as was shown in [21], is to add Gauss-Bonnet terms to the bulk action, which allows one to both put arbitrary types of matter and recover Einstein gravity on the brane. The result, as with the previous option, is that the exact cancellation between the brane tension and deficit angle is no longer imposed by the equations of motion.

Recently a third proposal [40, 41] was made, using the intersection of codimension-one branes rather than a simple "point-like" codimension-two brane as the place where matter is localized. However since both references include extra terms in the gravitational action, it is not clear whether it is the different geometry or the generalized action which allows one to generalize the brane equation of state.

Our view is that the above modifications are rather artificial because there is in general no reason to expect the metric to be well-behaved in the vicinity of a codimension-two or higher source. It appears to be an accident that the singularity is merely a conical one in the case of codimension-two source with pure tension. We will show below that the solution for a more general equation of state has more singular behaviour near the brane.

III. SETUP

As we have explained in the previous section, to allow for a general equation of state on a codimension-two brane with nonsingular solutions, it is necessary in Einstein gravity for the brane to have finite thickness. To study such a setup, we will model the branes using step function cores. This regularization scheme is similar to one that has been studied in the context of cosmic strings, (see, e.g. [49, 50]), with the difference that we will allow for the possibility of time dependent thickness. (This time dependence cannot be ruled out a priori for thick branes whose matter content itself is dynamical, as will be the case here). Furthermore, in order to find analytical solutions, we will treat the matter with $\rho \neq -p$ as a perturbation to a static pure-tension brane background. We will therefore write the metric ansatz as

$$ds^2 = -n(r,t)^2 dt^2 + a(r,t)^2 d\vec{x}^2 + b(r,t)^2 dr^2 + c(r,t)^2 d\theta^2 + 2E(r,t)drdt \tag{7}$$

with

$$n(r,t) = e^{N_0(r)+N_1(r,t)}; \qquad a(r,t) = a_0(t)e^{A_0(r)+A_1(r,t)}; \qquad b(r,t) = 1 + B_1(r,t);$$
$$c(r,t) = e^{C_0(r)+C_1(r,t)}; \qquad E(r,t) = E_1(r,t). \tag{8}$$

The bulk stress-energy content consists of a 6D cosmological constant Λ_6 and a two-form field F_{ab}, so that the bulk action can be written as[57]

$$S_{bulk} = \int d^6x \sqrt{-g}\left(\frac{\mathcal{R}}{16\pi G_6} - \Lambda_6 - \frac{1}{4}F^{ab}F_{ab}\right). \tag{9}$$

Although there exist solutions for both positive and negative values of Λ_6 [15], we will take it to be positive only for the rest of the paper. We assume that the only nonvanishing component of the vector potential is $A_\theta(r,t) = A_\theta^{(0)}(r) + A_\theta^{(1)}(r,t)$. For the sake of generality, we include a possible perturbation of the 6D cosmological constant, $\Lambda_6 \to \Lambda_6 + \delta\Lambda_6$. The background flux from the gauge field is necessary for getting a simple spherical compactification of the two extra dimensions. The full stress-energy tensor is taken to be of the form

$$T_b^a(r,t) = t_b^a(r,t) + \theta(r_0(t) - r)S_b^a(r,t) + \theta(r - r_*(t))S_{*b}^{\;a}(r,t). \tag{10}$$

Here, $r_0(t)$ and $r_*(t)$ are the radial positions of the edges of the thick 3-branes centered around the poles of the compact internal space, t_b^a refers to the bulk content, and S_b^a is the core stress energy, given by

$$
\begin{aligned}
S_t^t &= -\sigma - \rho; & S_x^x &= -\sigma + p; & S_r^r &= 0 + p_r^r; \\
S_\theta^\theta &= 0 + p_\theta^\theta; & S_t^r &= 0 + p_t^r; & S_r^t &= 0 + p_r^t; \\
S_{*t}^{\;t} &= -\sigma - \rho_*; & S_{*x}^{\;x} &= -\sigma + p_*; & S_{*r}^{\;r} &= 0 + p_{*r}^{\;r}; \\
S_{*\theta}^{\;\theta} &= 0 + p_{*\theta}^{\;\theta}; & S_{*t}^{\;r} &= 0 + p_{*t}^{\;r}; & S_{*r}^{\;t} &= 0 + p_{*r}^{\;t}.
\end{aligned} \tag{11}
$$

We treat the time dependence of the thickness as a perturbation, so that $r_0(t) = r_0 + \Delta r_0(t)$, $r_*(t) = r_* + \Delta r_*(t)$ and

$$
\begin{aligned}
\theta(r_0(t) - r) &= \theta(r_0 - r) + \delta(r_0 - r)\Delta r_0(t) + \mathcal{O}(\Delta r_0^2) & (12) \\
\theta(r - r_*(t)) &= \theta(r - r_*) - \delta(r - r_*)\Delta r_*(t) + \mathcal{O}(\Delta r_*^2). & (13)
\end{aligned}
$$

so that effectively, we can write the stress-energy tensor as

$$t_b^a + s_b^a + s_{*b}^{\;a} \tag{14}$$

with, e.g.,

$$
\begin{aligned}
s_t^t &= -\sigma\theta(r_0 - r) + [-\rho\theta(r_0 - r) - \sigma\delta(r_0 - r)\Delta r_0(t)] & (15) \\
s_i^i &= -\sigma\theta(r_0 - r) + [p\theta(r_0 - r) - \sigma\delta(r_0 - r)\Delta r_0(t)] & (16)
\end{aligned}
$$

and similarly for all other terms.

Here σ represents the tension of the regularized brane, and ρ, p represent contributions from ordinary matter on the standard-model (SM) brane, while starred quantities refer to matter on a hidden brane which is antipodal to the SM brane on the two-sphere bulk. All variables except σ are functions of both r and t. The appearance of off-diagonal terms in the perturbative stress-energy tensor might be surprising at first, but as we will explain, they must be included, since they cannot be made to vanish in a coordinate invariant manner.

The subscripts on the metric and gauge field perturbations indicate their order in a perturbative series in powers of ρ.[58] We will furthermore assume that time derivatives of the perturbations are of $\mathcal{O}(\rho^{3/2})$, which is implied by the usual law for conservation of energy $\dot{\rho} \sim (\dot{a}/a)\rho \sim \rho^{3/2}$. The framework is thus the same one that was used in the context of Randall-Sundrum cosmology in [52, 53].

IV. THE BACKGROUND

In this section we will generalize the static "football shaped" solution [13, 14] to the case of branes with nonvanishing thickness. (This will serve as the background around which we will later perturb by adding cosmological matter on the branes.) We will show that in the limit where the brane thickness vanishes, we recover the solution for delta function branes. This might at first seem like an obvious outcome, but in fact it does not necessarily happen, in the context of codimension-two objects in Einstein gravity [54]. In the present model, the extra dimensions are compactified on a two-sphere, with pure-tension branes positioned at the poles. The only effect of the branes on the bulk is to induce a deficit angle so that the internal space effectively looks like a sphere with a wedge taken out, hence the name "football-shaped" (the European reader may prefer to think in terms of a rugby ball). We will refer to the interior region of the thick brane as the "core," and the exterior as the bulk.

A. Core solutions

We first consider static solutions of the unperturbed Einstein and gauge field equations

$$8\pi G_6(\sigma + \Lambda_6) + 4\pi G_6 e^{-2C_0} A_\theta^{(0)'2} + 3A_0'' + C_0'' + 6A_0'^2 + C_0'^2 + 3A_0' C_0' = 0 \tag{17}$$

$$N_0'' - A_0'' + N_0'^2 - 3A_0'^2 + C_0'(N_0' - A_0') + 2A_0' N_0' = 0 \tag{18}$$

$$8\pi G_6 \Lambda_6 - 4\pi G_6 e^{-2C_0} A_\theta^{(0)'2} + 3A_0'(A_0' + N_0' + C_0') + C_0' N_0' = 0 \tag{19}$$

$$3A_0'' + N_0'' + 3A_0'^2 + N_0'^2 - C_0' N_0' - 3A_0' C_0' = 0 \tag{20}$$

$$\sigma(3A_0' + N_0') = 0 \tag{21}$$

$$A_\theta^{(0)''} + A_\theta^{(0)'}(3A_0' + N_0' - C_0') = 0 \tag{22}$$

in the interior (core) of the thick brane centered at $r = 0$. In the absence of any matter perturbation, conservation of energy implies eq. (21). Once this condition is imposed, combining the (rr) and $(\theta\theta)$ Einstein equations (18) and (20) gives $A_0' = N_0' = 0$, and the system of equations is solved by

$$A_0(r) = 0; \qquad N_0(r) = 0; \qquad e^{C_0(r)} = \frac{1}{k}\sin(kr);$$

$$A_\theta^{(0)}(r) = \frac{\beta}{k^2}\left(\cos(kr) - 1\right) \tag{23}$$

with

$$k^2 = 8\pi G_6(\sigma + \beta^2); \qquad \beta^2 = 2\Lambda_6. \tag{24}$$

In addition to this static solution, there exist solutions where the brane worldvolume is deSitter or anti-deSitter space if we relax the tuning between the gauge field and cosmological constant [15]. In the context of a related supersymmetric model, it has been shown [16, 45] that the flat brane solution is actually singled out by the dilaton equation of motion. Nothing forces us to choose the flat brane solution, but we do so for simplicity. It should also be noted that our background assumes both branes have the same tension. While there do exist solutions with different tensions on the two branes, these include warping and cannot be put in closed form, so the choice to expand around the simple unwarped background is also made for simplicity. Since our framework will allow us to perturb these two unnecessary but simplifying tunings (flat branes and equal tensions), our final results will not be affected by this choice.

We assume that the system is symmetric around the equator of the extra dimensions. Thus there is a brane also at the south pole, whose coordinate position is taken to be $r = \pi/k - \phi$. The solution in the core for the second brane is

$$A_0(r) = 0; \qquad N_0(r) = 0; \qquad e^{C_0(r)} = \frac{1}{k}\sin(k(r + \phi));$$

$$A_\theta^{(0)}(r) = \frac{\beta}{k^2}\left(\cos(k(r + \phi)) + 1\right) \tag{25}$$

where the difference in sign relative to the solution in the upper hemisphere is necessary for the gauge field to vanish at both poles. The phase ϕ is an integration constant to be determined by matching the core solutions smoothly to the bulk solutions, which we now consider.

B. Bulk solutions

In the bulk, the static solutions look like those for core,[59] except for the difference that $\sigma = 0$. The bulk solutions are therefore

$$e^{\bar{C}_0(r)} = \frac{1}{\bar{k}_1}\sin(\bar{k}(r + \bar{\phi})) \tag{26}$$

$$\bar{A}_\theta^{(0)}(r) = \bar{\beta}\frac{1}{\bar{k}_1\bar{k}}\cos(\bar{k}(r + \bar{\phi})) - \bar{\beta}_1 \tag{27}$$

Demanding that the metric and gauge field as well as their first derivatives be continuous across the junctions located at $r = r_0$ and $r = \pi/k - r_0 - \phi \equiv r_*$ (assuming that both branes have the same thickness), and using the fact that

$$\bar{\beta}^2 = \beta^2 = 2\Lambda_6; \qquad \bar{k}^2 = 8\pi G_6 \beta^2 \tag{28}$$

(c.f. eq. (24)), we find the following matching conditions from the $r = r_0$ junction:

$$\bar{k}_1^{-2} = \frac{\sin(kr_0)^2}{k^2} + \frac{\cos(kr_0)^2}{\bar{k}^2} \tag{29}$$

$$\tan(\bar{k}(r_0 + \bar{\phi})) = \frac{\bar{k}}{k} \tan(kr_0) \tag{30}$$

$$\bar{\beta}_1 = \frac{\beta}{k^2} \left[1 - \cos(kr_0) \left(1 - \frac{k^2}{\bar{k}^2} \right) \right]. \tag{31}$$

In the southern hemisphere, the bulk solution has the same form as for the northern one, except that a different constant of integration $\bar{\beta}_1^*$ must be used in place of $\bar{\beta}_1$ in the solution for the gauge field. At the $r = r_*$ junction we obtain

$$\bar{k}_1^{-2} = \frac{\sin(kr_0)^2}{k^2} + \frac{\cos(kr_0)^2}{\bar{k}^2} \tag{32}$$

$$\tan\left(\frac{\bar{k}}{k}\pi - \bar{k}r_0 - \bar{k}\phi + \bar{k}\bar{\phi} \right) = -\frac{\bar{k}}{k} \tan(kr_0) \tag{33}$$

$$\bar{\beta}_1^* = -\frac{\beta}{k^2} \left[1 - \cos(kr_0) \left(1 - \frac{k^2}{\bar{k}^2} \right) \right]. \tag{34}$$

The matching of the bulk gauge field solutions to both cores requires that $\bar{\beta}_1^* = -\bar{\beta}_1$, which makes the gauge field discontinuous at the equator. If the gauge field couples to matter, such a discontinuity is physically sensible only if the solutions in the two hemispheres are related to each other by a single-valued gauge transformation at the equator, which leads to the quantization condition (see [16, 55])

$$\bar{\beta}_1 = \frac{n}{2g} \tag{35}$$

where n is an integer and g is the $U(1)$ coupling constant.

Finally, we can use the relations (30) and (33) to show that

$$\phi = 2\bar{\phi} + \frac{\pi}{k} + \frac{m\pi}{\bar{k}} \tag{36}$$

where m is in principle an arbitrary integer, but the choice $m = -1$ ensures that both ϕ and $\bar{\phi}$ vanish when σ vanishes.

Let us now show that the usual relation between the deficit angle and the brane tension emerges in the $r_0 \to 0$ limit of the above solution. Since kr_0 remains finite, the brane tension σ scales like $1/r_0^2$. The effective four-dimensional tension is found by integrating over the volume of the extra dimensions (see Sec. VI):

$$\sigma^{(4)} = \frac{2\pi\sigma}{k^2} (1 - \cos(kr_0)) \tag{37}$$

so that as $r_0 \to 0$,

$$4G_6\sigma^{(4)} = (1 - \cos(kr_0)). \tag{38}$$

We must now consider how the deficit angle should be defined in the case we are considering, where the brane is thick. From the bulk point of view, the radial distance from the brane at $r = R$ is $R - r_0$. The circumference of a circle of radius R is $2\pi c(R, t)$, while the circumference of the brane is $2\pi c(r_0, t)$. If there is no matter on the brane, so that the internal space is perfectly spherical, we would expect that as $r_0 \to 0$ and $R \to 0$,

$$2\pi[c(R, t) - c(r_0, t)] = 2\pi(R - r_0). \tag{39}$$

On the other hand, if there is matter on the brane, it will modify the previous relation to read

$$2\pi(c(R, t) - c(r_0, t)) = 2\pi(R - r_0) \left(1 - \frac{\Delta}{2\pi} \right). \tag{40}$$

Thus we can define the deficit angle as

$$\Delta \equiv 2\pi \lim_{R \to 0} \left[\lim_{r_0 \to 0} 1 - \frac{c(R, t) - c(r_0, t)}{R - r_0} \right]. \tag{41}$$

To lowest order, this simply leads to

$$\Delta = 2\pi \left(1 - \lim_{R \to 0} \left[\lim_{r_0 \to 0} \left[\frac{\bar{k}_1^{-1} \sin(\bar{k}(R + \bar{\phi}))}{R}\right]\right]\right) \tag{42}$$

$$= 2\pi \left(1 - \lim_{R \to 0} \frac{\cos(kr_0) \sin(\bar{k}R)}{\bar{k} R}\right) \tag{43}$$

$$= 2\pi \left(1 - \cos(kr_0)\right) \tag{44}$$

The expected relation between the deficit angle and brane tension follows:

$$\Delta = 8\pi G_6 \sigma^{(4)} \tag{45}$$

We have therefore shown that this thick brane model consistently yields known results for a delta function brane in the limit of vanishing thickness. In other models of localized stress energy, it could happen that the (rr) and $(\theta\theta)$ components of the stress-energy tensor do not vanish in this limit, in contrast to the stress tensor for a brane. In such a case, the relation (45) would no longer be guaranteed [54]. Below we will demonstrate another means by which the relation (45) can be relaxed: adding brane matter with a different equation of state generalizes the form of (45). This is derived for thick branes, and it will be shown that the thin-brane limit is singular for these solutions. This is probably why departures from (45) have not been pointed out before, to our knowledge.

V.   PERTURBATIONS

It would be very interesting if an inflating solution, which started out with some relation between the brane tension and deficit angle differing from that required for a static solution, would dynamically relax to the static solution. The demonstration of such behaviour would be a convincing step toward solving the cosmological constant problem. It would also suggest that cosmology should be modified for ordinary matter on the brane—given that there is a drastic modification when the source on the brane corresponds to vacuum energy. To explore these issues, we want to add possibly time-dependent perturbations ρ and p to the stress-energy of the brane and see how the geometry responds. A physical example where this would be relevant is a phase transition on the brane, where part of the tension gets converted instantaneously into radiation.

Before considering the perturbed equations of motion, it is useful to find a set of coordinate invariant quantities associated with the metric perturbations in (7). Considering a small reparametrization of r and t such that

$$r \to r + f(r,t); \qquad t \to t + h(r,t), \tag{46}$$

where f and h are assumed to be of $\mathcal{O}(\rho)$, the perturbations transform as[60]

$$
\begin{aligned}
N_1 &\to N_1; \qquad A_1 \to A_1; \qquad B_1 \to B_1 + f'; \qquad C_1 \to C_1 + C_0'f; \\
A_\theta^{(1)} &\to A_\theta^{(1)} + A_\theta^{(0)\prime}f; \qquad E_1 \to E_1 - h' + \dot{f}; \qquad s_t^t \to s_t^t + \sigma\delta(r_0 - r)f; \\
s_i^i &\to s_i^i + \sigma\delta(r_0 - r)f; \qquad p_r^r \to p_r^r; \qquad p_\theta^\theta \to p_\theta^\theta; \qquad p_t^r \to p_t^r + \sigma\dot{f}; \\
p_r^t &\to p_r^t - \sigma h'.
\end{aligned} \tag{47}
$$

One can see why it is necessary to include off-diagonal terms in the perturbations to the core stress-energy tensor S_b^a, : they cannot be made to vanish in every frame. Moreover to have a consistent perturbative series in powers of ρ, E_1, p_r^r, p_r^t and h are $\mathcal{O}(\rho^{3/2})$, while all other terms are of order ρ.

From the form of the gauge transformations, we can deduce the following 11 gauge-invariant variables: [61]

$$Z = N_1' - A_1'; \qquad W = 3A_1' + N_1'; \qquad X = \frac{C_1'}{C_0'} - B_1 - \frac{C_0''}{C_0'^2}C_1;$$

$$Y = A_\theta^{(1)\prime} - A_\theta^{(0)\prime}(B_1 + C_1); \qquad U = \dot{A}_\theta^{(1)} - \frac{A_\theta^{(0)\prime}}{C_0'}\dot{C}_1;$$

$$\tilde{\rho} = \rho + \delta(r_0 - r)\left[\frac{\sigma}{C_0'}C_1 + \sigma\Delta r_0(t)\right]; \qquad \tilde{p} = p - \delta(r_0 - r)\left[\frac{\sigma}{C_0'}C_1 + \sigma\Delta r_0(t)\right];$$

$$\tilde{p}_t^r = p_t^r - \sigma\frac{\dot{C}_1}{C_0'}; \qquad \tilde{p}_r^t = p_r^t + p_r^r - \sigma E_1; \qquad \tilde{p}_5 = p_r^r; \qquad \tilde{p}_6 = p_\theta^\theta - p_r^r. \tag{48}$$

In terms of these variables, the equations of motion at first nontrivial order in the perturbation are[62]

$$Z' + C_0'Z = 2\left[\frac{\ddot{a}_0}{a_0} - \left(\frac{\dot{a}_0}{a_0}\right)^2\right] + 8\pi G(\tilde{\rho} + \tilde{p}) \tag{49}$$

$$W' - C_0'W = 8\pi G\tilde{p}_6 \tag{50}$$

$$Y' - C_0'Y - \beta e^{C_0}W = 0 \tag{51}$$

$$C_0'W + 8\pi G\beta e^{-C_0}Y = 3\left[\frac{\ddot{a}_0}{a_0} + \left(\frac{\dot{a}_0}{a_0}\right)^2\right] + 8\pi G(\tilde{p}_5 - \delta\Lambda_6) \tag{52}$$

$$C_0'X' + 2X\left(C_0'' + C_0'^2\right) + \frac{3}{2}C_0'W - 8\pi G\beta e^{-C_0}Y$$
$$= \frac{3}{2}\left[\frac{\ddot{a}_0}{a_0} + \left(\frac{\dot{a}_0}{a_0}\right)^2\right] - 2\pi G(\tilde{\rho} - 3\tilde{p} + 3\tilde{p}_6 + 4\delta\Lambda_6) \tag{53}$$

$$\tilde{p}_5' - C_0'\tilde{p}_6 + \sigma W = 0 \tag{54}$$

$$U' - \dot{Y} - \beta e^{C_0}\dot{X} = 0 \tag{55}$$

$$\tilde{p}_r^t = 0 \tag{56}$$

$$3\frac{\dot{a}_0}{a_0}Z - C_0'\dot{X} + \frac{3}{4}(\dot{Z} - \dot{W}) = 8\pi G(\tilde{p}_t^r - \beta e^{-C_0}U) \tag{57}$$

$$\dot{\tilde{\rho}} + 3\frac{\dot{a}_0}{a_0}(\tilde{\rho} + \tilde{p}) = \tilde{p}_t^{r\,\prime} + C_0'\tilde{p}_t^r + \sigma\dot{X} \tag{58}$$

These are not all independent however. It can be shown that eq. (51) and eq. (55) can be obtained as combinations of the other equations in the system, and can therefore be ignored. We have found the general solution to this reduced system of equations in appendix A.

The general solution to (49-58) is complicated, but we are principally interested in just one aspect, the Friedmann equations which describe the expansion of the universe for an observer on one of the branes. In the present construction, the Friedmann equations arise through imposing the appropriate boundary conditions at the junctions of the regions interior and exterior to the branes. In terms of our gauge-invariant variables, continuity of the functions but not of the derivatives is required at the interfaces, since the equations of motion do not contain any spatial second derivatives. Moreover we require that all functions be nonsingular at the poles. Finally, in what follows, we assume that $\tilde{\rho}, \tilde{p}, \tilde{\rho}_*$, $\tilde{p}_*$ are functions of time only and that $\tilde{p}_6 = e^{2C_0(r)}\mathcal{P}_6(t)$ and $\tilde{p}_{*6} = e^{2C_0(r)}\mathcal{P}_{*6}(t)$.[63]

With this ansatz for the brane energy density, applying boundary conditions allows us to derive the Friedmann equations. However, there is one last point to address before writing them down: the 6D quantities $\tilde{\rho}$ and $\tilde{p}$ we have used so far are not the ones an observer on the brane would identify as the energy density and pressure. It is therefore necessary to define what the effective energy density and pressure are for a 4D observer if we want to compare the Friedmann equations in this model to the standard ones.

VI. EFFECTIVE FOUR DIMENSIONAL QUANTITIES

To define the effective four dimensional quantities, we integrate the 6D quantities over the thickness of the brane,

$$S^{(4)\,a}_{\quad\ b} = 2\pi \int_0^{r_0} dr\, b(r,t)\, c(r,t)\, S^{(6)\,a}_{\quad\ b} \tag{59}$$

which perturbatively leads to

$$\sigma^{(4)} + \rho^{(4)} = 2\pi \int_0^{r_0} dr\, e^{C_0}(1 + B_1 + C_1)(-s_t^t) \tag{60}$$

$$-\sigma^{(4)} + p^{(4)} = 2\pi \int_0^{r_0} dr\, e^{C_0}(1 + B_1 + C_1)(s_i^i) \tag{61}$$

$$p_5^{(4)} = 2\pi \int_0^{r_0} dr\, e^{C_0}\tilde{p}_5 \tag{62}$$

$$p_6^{(4)} = 2\pi \int_0^{r_0} dr\, e^{C_0}\tilde{p}_6. \tag{63}$$

and similarly for the other brane.[64] In order to compute $\rho^{(4)}$ and $p^{(4)}$, we choose a gauge and find C_1 or B_1 explicitly. For this purpose it is convenient to work in coordinates where $B_1 = E_1 = 0$. From the form of the gauge transformations (47), it can be shown that the 4D quantities are gauge invariant. For example, $\delta(\sigma^{(4)} + \rho^{(4)}) = 2\pi \int_0^{r_0} dr \frac{d}{dr}(e^{C_0}\sigma f)$ is a surface term which vanishes because $f = 0$ at $r = 0$ and $\sigma = 0$ at $r = r_0^+$.

Similarly, the 4D Newton constant is related to the 6D one by dimensional reduction,

$$G_6 = G_4 \times V \tag{64}$$

$$= G_4 \int_0^{2\pi} d\theta \int_0^{\pi/k-\phi} e^{C_0(r)} \tag{65}$$

$$= G_4 \times \frac{4\pi}{k^2 \bar{k}^2}(\bar{k}^2 + (k^2 - \bar{k}^2)\cos(kr_0)) \tag{66}$$

where we neglect corrections of $\mathcal{O}(\rho)$.

After gauge-fixing, the Friedmann equations and conservation of energy can be rewritten in terms of the effective 4D quantities (see the appendix for the full solutions)

$$\left(\frac{\dot{a}_0}{a_0}\right)^2 = \frac{8\pi G_4}{3}\left(\rho^{(4)} + \rho_*^{(4)} + \Lambda_{\text{eff}}\right) \tag{67}$$

$$\frac{\ddot{a}_0}{a_0} = \left(\frac{\dot{a}_0}{a_0}\right)^2 - 4\pi G_4\left(\rho^{(4)} + p^{(4)} + \rho_*^{(4)} + p_*^{(4)}\right) \tag{68}$$

$$\dot{\rho}^{(4)} = -3\frac{\dot{a}_0}{a_0}\left(\rho^{(4)} + p^{(4)}\right) \tag{69}$$

$$\dot{\rho}_*^{(4)} = -3\frac{\dot{a}_0}{a_0}\left(\rho_*^{(4)} + p_*^{(4)}\right). \tag{70}$$

These constitute the main result of this paper, and show that we indeed recover standard cosmology on the brane, except for the constant of integration Λ_{eff} which represents bulk contributions to the 4D cosmological constant, and the contribution of "dark matter" which is on the second brane. Of course we also expect corrections of order ρ^2 and higher which would appear at higher order in the perturbation expansion [53], but these are not relevant for the immediate question concerning self-tuning.

VII. DISCUSSION

To elucidate our results (67-70), let us first explain the role of the constant Λ_{eff}. It might at first be surprising that our solutions contains a term that does not vanish when matter perturbations are absent. However, this simply reflects the fact that our choice to perturb around a static background solution was arbitrary, and there exist evolving solutions even in the absence of matter perturbations. This is because we are free to choose any set of values for the parameters of the background solution $\sigma^{(4)}$, β and Λ_6.[65] We arbitrarily tuned $\beta^2 = 2\Lambda_6$ to ensure that the brane does not expand. Solutions with nonzero Λ_{eff} simply correspond to using a different set of values for the background parameters, for which the tuning between the field strength and bulk cosmological constant has been perturbed. The value of the constant Λ_{eff} is thus determined by the background around which matter perturbations are being added, and cannot be used to subsequently tune the resulting solutions.

We will now consider some limiting cases which provide a consistency check on our solutions, and which illustrate the differences between the thick-brane model and its thin-brane limit.

A. Thin brane limit

First we would like to consider how our results behave as we take the thin brane (*i.e.*, $r_0 \to 0$) limit. Earlier we argued that for a delta function brane with an equation of state differing from pure tension, the 4D components of the metric have singular behaviour at the origin. Let us explain how this is manifested in our results.

The thin brane limit corresponds to taking $k \to \infty$, since the constant kr_0 remains finite. This follows from the requirement that the effective four dimensional quantities $\sigma^{(4)}$, $\rho^{(4)}$ and $p^{(4)}$ should be held fixed as we vary the brane thickness. We can then see from our results why the $r_0 \to 0$ limit produces singular results at the branes unless the

matter on the brane consists of pure tension. From the metric perturbations A_1 and N_1, we find that[66]

$$A_1'(r_0, t) = -k \frac{G_6 \left(\rho^{(4)} + p^{(4)} \right)}{\sin(kr_0)} + \mathcal{O}\left(1/k \right) \tag{71}$$

$$N_1'(r_0, t) = 3k \frac{G_6 \left(\rho^{(4)} + p^{(4)} \right)}{\sin(kr_0)} + \mathcal{O}\left(1/k \right) \tag{72}$$

where we have concentrated only on the brane located at the upper pole, but would of course find similar results for the other brane. These diverge in the thin brane limit (since $k \sim 1/r_0$ as $r_0 \to 0$) unless the perturbations satisfy $p^{(4)} = -\rho^{(4)}$. Thus it is only possible to have pure tension on a codimension-two delta function brane, if we (unreasonably) require that g_{tt} and g_{xx} be regular at the brane. On the other hand if we allow an arbitrary equation of state, then A_1' and N_1' diverge like $1/r$ near the brane, indicating that A_1 and N_1 go like $\ln r$.

We stress again that regularity at the brane is probably too much to ask of a solution when a delta function source is present; rather one should expect singularities in this case. Therefore, we do not interpret our results as indicating any fundamental obstruction to having general types of matter on a codimension-two delta function brane. Rather, eqs. (71-72) imply the mildly singular behaviour

$$a(r, t) \sim r^{-G_6(\rho+p)}, \quad n(r, t) \sim r^{3G_6(\rho+p)} \tag{73}$$

for the 4D metric components in the vicinity of the brane.

B. Deficit angle

We turn our attention now to the deficit angle which, as was previously mentioned, is defined by

$$\Delta \equiv 2\pi \lim_{R \to 0} \left[\lim_{r_0 \to 0} 1 - \frac{c(R, t) - c(r_0, t)}{R - r_0} \right] \tag{74}$$

around the brane located at $r = 0$. Plugging our solutions $c(r, t) \approx e^{C_0(r)}(1 + C_1(r, t))$ into this, we obtain

$$\Delta = 2\pi G_6 (4\sigma^{(4)} + \rho^{(4)} - 3p^{(4)}) \tag{75}$$

The same procedure performed around the other brane leads to

$$\Delta_* = 2\pi G_6 (4\sigma^{(4)} + \rho_*^{(4)} - 3p_*^{(4)}) \tag{76}$$

We see that the deficit angle indeed responds to changes in the stress-energy on the brane; moreover in the familiar case where $p^{(4)} = -\rho^{(4)}$ (pure tension), the relation reduces to its expected form $\Delta = 8\pi G_6 \sigma_{tot}^{(4)}$ where $\sigma_{tot}^{(4)} = \sigma^{(4)} + \rho^{(4)}$. However, this is not enough to insure a static solution, as can be seen from eq. (67). At first this looks quite mysterious: in the pure tension case, why is the effect of the tension not canceled by the conical singularity, as was the case for the unperturbed solution? The answer, given in the next subsection, has to do with the effect of the perturbation on the bulk fields, in particular the gauge field.

C. General equations of state

With our results we can now answer the physical question posed at the beginning of section 5: suppose we start from a static solution which undergoes a phase transition that converts a part of the brane's tension into radiation—how does the geometry respond to this sudden change in the equation of state? The perturbation to the stress tensor can be thought of as the sum of two pieces,

$$\rho = \delta\sigma + \rho_{\text{rad}}$$
$$p = -\delta\sigma + \frac{1}{3}\rho_{\text{rad}} \tag{77}$$

where $\delta\sigma$ is negative, and initially $\rho = 0$. According to the Friedmann equations, the universe will start to collapse toward a big crunch, as one would expect for an anti-deSitter solution; the universe behaves just as though it had a negative cosmological constant. The conclusion actually does not require the transition to be sudden, nor does it have

to produce radiation. The same outcome occurs even if the equation of state of the perturbation changes arbitrarily slowly from $w = -1$ to any larger value.

For example we could consider such a transition in a more smooth way by choosing a positive initial value of $\delta\rho$, tuning the solution (using the integration constant Λ_{eff}) to be initially static, and imposing a time-dependent equation of state for the perturbation, $w = (-1 + \tanh \epsilon t)/2$. This represents the gradual conversion of the perturbation from vacuum energy in the past to pressureless matter in the future. Again, the Friedmann equations predict a collapsing universe due to the effectively negative cosmological constant at late times.

Similarly, if we start from an initially static solution with a negative value of $\delta\rho$ and the opposite time dependence for the equation of state, $w = (-1 - \tanh \epsilon t)/2$, we obtain an inflating solution at late times due to the positive change in the brane tension. In all cases, it is clear that no self-tuning mechanism cancels the effect of the change in tension.

Now we address the seeming paradox: how can our results be self-consistent? We find that the Hubble rate is sensitive to changes in the energy density on the brane, yet the deficit angle tracks this energy density in the way which we would hope for if there was self-tuning of the 4D cosmological constant to zero. The key point is that the tuning between the deficit angle and brane tension is not sufficient to lead to a static solution; one also needs the field strength and bulk cosmological constant to be tuned. In general, $\delta\rho$ induces a perturbation to the field strength $F^2 \cong F_0^2 + 2F_0 \delta F$, where in terms of our variables, we find that

$$\delta F = \sqrt{2} e^{-C_0} Y(r,t) \tag{78}$$

In the static solution, the field strength had to be tuned relative to Λ_6. Apparently it is this tuning which is spoiled in the presence of matter on the brane, rather than anything related to the deficit angle.

VIII.   SUMMARY AND OUTLOOK

In this paper we constructed a thick codimension-two braneworld model in order to determine whether such scenarios have a self-tuning mechanism which could help solve the cosmological constant problem, and also to investigate the question of whether they allow us to recover standard cosmology.

The good news is that the second question is answered in the affirmative. Codimension-two braneworlds are quite consistent with the general expectations from decoupling and dimensional reduction: the 4D effective theory looks like general relativity at lowest order in the density of matter on the branes. Although corrections to GR will undoubtedly appear at order ρ^2, we have not tried to compute these in the present work.

On the other hand, we find an obstacle to self-tuning of the effective 4D cosmological constant in this model. Sarting from a static solution, any phase transition which changes the tension of the brane will either lead to a contracting or an expanding universe. In the second case, once the matter has redshifted away, we do not recover the original static solution, but rather a deSitter one, where the difference relative to the original static solution comes from the fact that the phase transition has spoiled the original tuning between the bulk gauge field strength and the bulk cosmological constant.

We were motivated to study a thick brane model because of the apparent impossibility of putting matter with a general equation of state on a codimension-two brane in Einstein gravity. We find that there is actually no prohibition; instead the metric becomes singular near the brane when it contains general kinds of matter. In retrospect this is not surprising, and the fact that pure-tension branes induce only a conical singularity appears to be fortuitous. Accepting the singular nature of the metric, we derived an expression for the deficit angle with arbitrary matter on the brane,

$$\Delta = 2\pi G_6 \left(4\sigma^{(4)} + \rho^{(4)} - 3p^{(4)} \right) \tag{79}$$

which reduces to the expected one for a pure-tension brane.

Although our results show that codimension-two braneworlds in Einstein gravity don't solve the cosmological constant problem, the outlook may be better in models where there is supersymmetry in the bulk [16, 45, 46]. The encouraging point in our results is that the deficit angle *does* respond to arbitrary changes in the brane stress energy in such a way as to cancel its contribution (through the singular part of the gravitational action). What is needed is a symmetry to maintain a vanishing bulk contribution to the effective 4D cosmological constant, and it has been argued that supersymmetry can do precisely this. In SUSY models, the dilaton equation of motion insures that the tuning between the gauge field and bulk cosmological constant (which appears in the guise of a scalar field potential) required for a static solution always holds as long as the dilaton is stabilized. Since we have shown that it is the detuning between the field strength and bulk cosmological constant that leads to non-static solutions in Einstein gravity, it seems reasonable to expect that in models with supersymmetry, there will indeed be a self-tuning mechanism to cancel the effective cosmological constant. Checking this explicitly is the subject of ongoing work.

IX. ACKNOWLEDGEMENTS

APPENDIX A: PERTURBATIVE SOLUTIONS

We present here general solutions to the perturbed equations of motion (49-58). The superscripts I, II, III and IV are meant to identify quantites associated with the different regions in the model, namely the upper core ($r = 0$ to $r = r_0$), the upper bulk ($r = r_0$ to $r = \pi/2/\bar{k} - \bar{\phi}$), lower bulk ($r = \pi/2/\bar{k} - \bar{\phi}$ to $r = \pi/k - \phi - r_0$) and lower core ($r = \pi/k - \phi - r_0$ to $r = \pi/k - \phi$). We have defined $\bar{r} \equiv r + \bar{\phi}$ and $\hat{r} \equiv r + \phi$. We have not written the solutions for the gauge invariant variable $U(r,t)$, since it can be obtained trivially from the algebraic equation (57) once the other variables have been solved for.

$$Z^I = \frac{\mathcal{F}_1^I(t)}{\sin(kr)} - 2\frac{\cos(kr)}{k\sin(kr)}\left[4\pi G_6(\rho + p) + \frac{\ddot{a}_0}{a_0} - \left(\frac{\dot{a}_0}{a_0}\right)^2\right] \tag{A1}$$

$$W^I = \mathcal{F}_2^I(t)\sin(kr) - \frac{8\pi G_6 \sin(kr)\cos(kr)}{k^3}\mathcal{P}_6 \tag{A2}$$

$$\tilde{p}_5 = \mathcal{F}_3^I(t) + \frac{\sigma\cos(kr)}{k}\mathcal{F}_2^I(t) - \frac{(k^2 + 8\pi G_6\sigma)(\cos(kr)^2 - \sin(kr)^2)}{4k^4}\mathcal{P}_6 \tag{A3}$$

$$Y^I = \frac{(k^2 + 8\pi G_6\sigma + 2(k^2 - 8\pi G_6\sigma)\cos(kr)^2)\sin(kr)}{4k^5\beta}\mathcal{P}_6$$

$$- \frac{(k^2 - 8\pi G_6\sigma)\sin(kr)\cos(kr)}{8\pi G_6\beta k^2}\mathcal{F}_2^I(t)$$

$$+ \frac{\sin(kr)}{8\pi G_6\beta k}\left[3\frac{\ddot{a}_0}{a_0} + 3\left(\frac{\dot{a}_0}{a_0}\right)^2 + 8\pi G_6\mathcal{F}_3^I(t) - 8\pi G_6\delta\Lambda_6\right] \tag{A4}$$

$$X^I = \frac{\mathcal{F}_4^I(t)}{\cos(kr)^2} + \frac{(5k^2 - 16\pi G_6\sigma)\cos(kr)}{6k^3}\mathcal{F}_2^I(t)$$

$$+ \frac{\pi G_6}{16k^6\cos(kr)^2}\left[17k^2 + 16\pi G_6\sigma + 32\cos(kr)^2(k^2 - 4\pi G_6\sigma)\right.$$

$$- 8\cos(kr)^4(11k^2 - 16\pi G_6\sigma)\right]\mathcal{P}_6$$

$$- \frac{(2\cos(kr)^2 - 1)}{8k^2\cos(kr)^2}\left[9\frac{\ddot{a}_0}{a_0} + 9\left(\frac{\dot{a}_0}{a_0}\right)^2 - 4\pi G_6(\rho - 3p + 8\delta\Lambda_6 - 4\mathcal{F}_3^I(t))\right] \tag{A5}$$

$$\tilde{p}_t^r = -3\frac{\cos(kr)}{k\sin(kr)}\frac{\dot{a}_0}{a_0}(\rho + p) + \frac{d}{dt}\left[-\frac{9\sigma(1 + 2\cos(kr)^2)}{8k^3\sin(kr)\cos(kr)}\left(\frac{\ddot{a}_0}{a_0} + \left(\frac{\dot{a}_0}{a_0}\right)^2\right)\right.$$

$$+ \frac{\sigma\pi G_6 - 2\cos(kr)^2(k^2 - \sigma\pi G_6)}{2k^3\sin(kr)\cos(kr)}\rho - \frac{\sigma\pi G_6(1 + 2\cos(kr)^2)}{k^3\sin(kr)\cos(kr)}\left(\frac{3}{2}p + 2\mathcal{F}_3^I(t)\right)$$

$$+ \frac{\sigma\pi G_6}{48k^7\sin(kr)\cos(kr)}\left\{-3(17k^2 + 16\sigma\pi G_6) + 96\cos(kr)^2(k^2 - 4\sigma\pi G_6)\right.$$

$$- 8\cos(kr)^4(11k^2 - 16\sigma\pi G_6)\right\}\mathcal{P}_6$$

$$+ \left.\frac{(5k^2 - 16\pi G_6\sigma)\sigma\cos(kr)^2}{12k^4\sin(kr)}\mathcal{F}_2^I(t) - \frac{\sigma}{k\sin(kr)\cos(kr)}\mathcal{F}_4^I(t)\right] + \frac{\mathcal{F}_5^I(t)}{\sin(kr)} \tag{A6}$$

$$Z^{II} = \frac{\mathcal{F}_1^{II}(t)}{\sin(\bar{k}\bar{r})} - 2\frac{\cos(\bar{k}\bar{r})}{\bar{k}\sin(\bar{k}\bar{r})}\left[\frac{\ddot{a}_0}{a_0} - \left(\frac{\dot{a}_0}{a_0}\right)^2\right] \tag{A7}$$

$$W^{II} = \mathcal{F}_2^{II}(t)\sin(\bar{k}\bar{r}) \tag{A8}$$

$$Y^{II} = -\frac{\bar{k}\sin(\bar{k}\bar{r})\cos(\bar{k}\bar{r})}{8\pi G_6\beta\bar{k}_1}\mathcal{F}_2^{II}(t) + \frac{\sin(\bar{k}\bar{r})}{8\pi G_6\beta\bar{k}_1}\left[3\frac{\ddot{a}_0}{a_0} + 3\left(\frac{\dot{a}_0}{a_0}\right)^2 - 8\pi G_6\delta\Lambda_6\right] \tag{A9}$$

$$X^{II} = \frac{5\cos(\bar{k}\bar{r})}{6\bar{k}}\mathcal{F}_2^{II}(t) - \frac{(2\cos(\bar{k}\bar{r})^2 - 1)}{8\bar{k}^2\cos(\bar{k}\bar{r})^2}\left[9\frac{\ddot{a}_0}{a_0} + 9\left(\frac{\dot{a}_0}{a_0}\right)^2 - 32\pi G_6\delta\Lambda_6\right]$$
$$+ \frac{\mathcal{F}_4^{II}(t)}{\cos(\bar{k}\bar{r})^2} \tag{A10}$$

$$Z^{III} = \frac{\mathcal{F}_1^{III}(t)}{\sin(\bar{k}\bar{r})} - 2\frac{\cos(\bar{k}\bar{r})}{\bar{k}\sin(\bar{k}\bar{r})}\left[\frac{\ddot{a}_0}{a_0} - \left(\frac{\dot{a}_0}{a_0}\right)^2\right] \tag{A11}$$

$$W^{III} = \mathcal{F}_2^{III}(t)\sin(\bar{k}\bar{r}) \tag{A12}$$

$$Y^{III} = -\frac{\bar{k}\sin(\bar{k}\bar{r})\cos(\bar{k}\bar{r})}{8\pi G_6\beta\bar{k}_1}\mathcal{F}_2^{III}(t) + \frac{\sin(\bar{k}\bar{r})}{8\pi G_6\beta\bar{k}_1}\left[3\frac{\ddot{a}_0}{a_0} + 3\left(\frac{\dot{a}_0}{a_0}\right)^2 - 8\pi G_6\delta\Lambda_6\right] \tag{A13}$$

$$X^{III} = \frac{5\cos(\bar{k}\bar{r})}{6\bar{k}}\mathcal{F}_2^{III}(t) - \frac{(2\cos(\bar{k}\bar{r})^2 - 1)}{8\bar{k}^2\cos(\bar{k}\bar{r})^2}\left[9\frac{\ddot{a}_0}{a_0} + 9\left(\frac{\dot{a}_0}{a_0}\right)^2 - 32\pi G_6\delta\Lambda_6\right]$$
$$+ \frac{\mathcal{F}_4^{III}(t)}{\cos(\bar{k}\bar{r})^2} \tag{A14}$$

$$Z^{IV} = \frac{\mathcal{F}_1^{IV}(t)}{\sin(k\hat{r})} - 2\frac{\cos(k\hat{r})}{k\sin(k\hat{r})}\left[4\pi G_6(\rho_* + p_*) + \frac{\ddot{a}_0}{a_0} - \left(\frac{\dot{a}_0}{a_0}\right)^2\right] \tag{A15}$$

$$W^{IV} = \mathcal{F}_2^{IV}(t)\sin(k\hat{r}) - \frac{8\pi G_6\sin(k\hat{r})\cos(k\hat{r})}{k^3}\mathcal{P}_{*6} \tag{A16}$$

$$\tilde{p}_{*5} = \mathcal{F}_3^{IV}(t) + \frac{\sigma\cos(k\hat{r})}{k}\mathcal{F}_2^{IV}(t) - \frac{(k^2 + 8\pi G_6\sigma)(\cos(k\hat{r})^2 - \sin(k\hat{r})^2)}{4k^4}\mathcal{P}_{*6} \tag{A17}$$

$$Y^{IV} = \frac{(k^2 + 8\pi G_6\sigma + 2(k^2 - 8\pi G_6\sigma)\cos(k\hat{r})^2)\sin(k\hat{r})}{4k^5\beta}\mathcal{P}_{*6}$$
$$- \frac{(k^2 - 8\pi G_6\sigma)\sin(k\hat{r})\cos(k\hat{r})}{8\pi G_6\beta k^2}\mathcal{F}_2^{IV}(t)$$
$$+ \frac{\sin(k\hat{r})}{8\pi G_6\beta k}\left[3\frac{\ddot{a}_0}{a_0} + 3\left(\frac{\dot{a}_0}{a_0}\right)^2 + 8\pi G_6\mathcal{F}_3^{IV}(t) - 8\pi G_6\delta\Lambda_6\right] \tag{A18}$$

$$X^{IV} = \frac{\mathcal{F}_4^{IV}(t)}{\cos(k\hat{r})^2} + \frac{(5k^2 - 16\pi G_6\sigma)\cos(k\hat{r})}{6k^3}\mathcal{F}_2^{IV}(t)$$
$$+ \frac{\pi G_6}{16k^6\cos(k\hat{r})^2}\left[17k^2 + 16\pi G_6\sigma + 32\cos(k\hat{r})^2(k^2 - 4\pi G_6\sigma)\right.$$
$$- \left. 8\cos(k\hat{r})^4(11k^2 - 16\pi G_6\sigma)\right]\mathcal{P}_{*6}$$
$$- \frac{(2\cos(k\hat{r})^2 - 1)}{8k^2\cos(k\hat{r})^2}\left[9\frac{\ddot{a}_0}{a_0} + 9\left(\frac{\dot{a}_0}{a_0}\right)^2 - 4\pi G_6(\rho_* - 3p_* + 8\delta\Lambda_6 - 4\mathcal{F}_3^{IV}(t))\right] \tag{A19}$$

$$\tilde{p}_{*t}^{\ r} = -3\frac{\cos(k\hat{r})}{k\sin(k\hat{r})}\frac{\dot{a}_0}{a_0}(\rho_* + p_*) + \frac{d}{dt}\left[-\frac{9\sigma(1 + 2\cos(k\hat{r})^2)}{8k^3\sin(k\hat{r})\cos(k\hat{r})}\left(\frac{\ddot{a}_0}{a_0} + \left(\frac{\dot{a}_0}{a_0}\right)^2\right)\right.$$
$$+ \frac{\sigma\pi G_6 - 2\cos(k\hat{r})^2(k^2 - \sigma\pi G_6)}{2k^3\sin(k\hat{r})\cos(k\hat{r})}\rho_* - \frac{\sigma\pi G_6(1 + 2\cos(k\hat{r})^2)}{k^3\sin(k\hat{r})\cos(k\hat{r})}\left(\frac{3}{2}p_* + 2\mathcal{F}_3^{IV}(t)\right)$$
$$+ \frac{\sigma\pi G_6}{48k^7\sin(k\hat{r})\cos(k\hat{r})}\left\{-3(17k^2 + 16\sigma\pi G_6) + 96\cos(k\hat{r})^2(k^2 - 4\sigma\pi G_6)\right.$$
$$- \left.\left. 8\cos(k\hat{r})^4(11k^2 - 16\sigma\pi G_6)\right\}\mathcal{P}_{*6}\right.$$

$$+ \frac{(5k^2 - 16\pi G_6\sigma)\sigma\cos(k\hat{r})^2}{12k^4\sin(k\hat{r})}\mathcal{F}_2^{IV}(t) - \frac{\sigma}{k\sin(k\hat{r})\cos(k\hat{r})}\mathcal{F}_4^{IV}(t) \Big] + \frac{\mathcal{F}_5^{IV}(t)}{\sin(k\hat{r})} \tag{A20}$$

These solutions solve the equations of motion (49-58). However, we still need to impose boundary conditions.

Before doing so, we will specialize to a gauge where $E_1 = B_1 = 0$. We can then use the above results to find $C_1(r,t)$ and $A_\theta^{(1)}(r,t)$. (The explicit solutions for $A_1(r,t)$ and $N_1(r,t)$ are of no particular interest, so we won't bother writing them explicitly).

$$
\begin{aligned}
C_1^I =&\ \mathcal{F}_4^I(t) + \frac{k\cos(kr)}{\sin(kr)}\mathcal{F}_6^I(t) + \frac{(5k^2 - 16\sigma\pi G_6)\cos(kr)}{6k^3}\mathcal{F}_2^I(t) \\
&+ \frac{\pi G_6}{16k^6\sin(kr)}\Big[\sin(kr)(17k^2 + 16\sigma\pi G_6 - 4\cos(kr)^2(11k^2 - 16\sigma\pi G_6)) \\
&- 4(3k^2 + 16\sigma\pi G_6)kr\cos(kr)\Big]\mathcal{P}_6 \\
&- \frac{2kr\cot(kr) - 1}{8k^2}\left[9\frac{\ddot{a}_0}{a_0} + 9\left(\frac{\dot{a}_0}{a_0}\right)^2 - 4\pi G_6(\rho - 3p + 8\delta\Lambda_6 - 4\mathcal{F}_3^I(t))\right]
\end{aligned} \tag{A21}
$$

$$
\begin{aligned}
C_1^{II} =&\ \mathcal{F}_4^{II}(t) + \frac{\bar{k}\cos(\bar{k}\bar{r})}{\sin(\bar{k}\bar{r})}\mathcal{F}_6^{II}(t) + \frac{5\cos(\bar{k}\bar{r})}{6\bar{k}}\mathcal{F}_2^{II}(t) \\
&- \frac{2\bar{k}r\cot(\bar{k}\bar{r}) - 1}{8\bar{k}^2}\left[9\frac{\ddot{a}_0}{a_0} + 9\left(\frac{\dot{a}_0}{a_0}\right)^2 - 32\pi G_6\delta\Lambda_6\right]
\end{aligned} \tag{A22}
$$

$$
\begin{aligned}
C_1^{III} =&\ \mathcal{F}_4^{III}(t) + \frac{\bar{k}\cos(\bar{k}\bar{r})}{\sin(\bar{k}\bar{r})}\mathcal{F}_6^{III}(t) + \frac{5\cos(\bar{k}\bar{r})}{6\bar{k}}\mathcal{F}_2^{III}(t) \\
&- \frac{2\bar{k}r\cot(\bar{k}\bar{r}) - 1}{8\bar{k}^2}\left[9\frac{\ddot{a}_0}{a_0} + 9\left(\frac{\dot{a}_0}{a_0}\right)^2 - 32\pi G_6\delta\Lambda_6\right]
\end{aligned} \tag{A23}
$$

$$
\begin{aligned}
C_1^{IV} =&\ \mathcal{F}_4^{IV}(t) + \frac{k\cos(k\hat{r})}{\sin(k\hat{r})}\mathcal{F}_6^{IV}(t) + \frac{(5k^2 - 16\sigma\pi G_6)\cos(k\hat{r})}{6k^3}\mathcal{F}_2^{IV}(t) \\
&+ \frac{\pi G_6}{16k^6\sin(k\hat{r})}\Big[\sin(k\hat{r})(17k^2 + 16\sigma\pi G_6 - 4\cos(k\hat{r})^2(11k^2 - 16\sigma\pi G_6)) \\
&- 4((3k^2 + 16\sigma\pi G_6)kr + (11k^2 - 16\sigma\pi G_6)k\phi)\cos(k\hat{r})\Big]\mathcal{P}_{*6} \\
&- \frac{2kr\cot(k\hat{r}) - 1}{8k^2}\left[9\frac{\ddot{a}_0}{a_0} + 9\left(\frac{\dot{a}_0}{a_0}\right)^2 - 4\pi G_6(\rho_* - 3p_* + 8\delta\Lambda_6 - 4\mathcal{F}_3^{IV}(t))\right]
\end{aligned} \tag{A24}
$$

$$
\begin{aligned}
A_\theta^{(1)I} =&\ \mathcal{F}_7^I(t) - \frac{\beta\sin(kr)}{k}\mathcal{F}_6^I(t) + \frac{\beta\cos(kr)}{k^2}\mathcal{F}_4^I(t) \\
&+ \frac{(3k^2(k^2 - 8\sigma\pi G_6) + 4\beta^2\pi G_6(5k^2 - 16\sigma\pi G_6))\cos(kr)^2}{48\pi G_6\beta k^5}\mathcal{F}_2^I(t) \\
&+ \frac{(4\beta^2\pi G_6 kr\sin(kr) - \cos(kr)(k^2 - 6\beta^2\pi G_6))}{k^4\beta}\mathcal{F}_3^I(t) \\
&+ \frac{1}{48k^8\beta}\Big[(-8k^2(k^2 - 8\sigma\pi G_6) - 4\beta^2\pi G_6(11k^2 - 16\sigma\pi G_6))\cos(kr)^3 \\
&- (12k^2(k^2 + 8\sigma\pi G_6) - 3\beta^2\pi G_6(29k^2 + 80\sigma\pi G_6)))\cos(kr) \\
&+ 12\pi G_6\beta^2(3k^2 + 16\sigma\pi G_6)kr\sin(kr)\Big]\mathcal{P}_6(t) \\
&+ \frac{3(6\beta^2\pi G_6 kr\sin(kr) - \cos(kr)(k^2 - 9\beta^2\pi G_6))}{8k^4\beta\pi G_6}\left[\frac{\ddot{a}_0}{a_0} + \left(\frac{\dot{a}_0}{a_0}\right)^2\right] \\
&- \frac{\beta\pi G_6(2kr\sin(kr) + 3\cos(kr))}{2k^4}(\rho - 3p) \\
&- \frac{8\beta^2\pi G_6 kr\sin(kr) - \cos(kr)(k^2 - 12\beta^2\pi G_6)}{k^4\beta}\delta\Lambda_6
\end{aligned} \tag{A25}
$$

$$
A_\theta^{(1)II} = \mathcal{F}_7^{II}(t) - \frac{\beta\sin(\bar{k}\bar{r})}{\bar{k}_1}\mathcal{F}_6^{II}(t) + \frac{\beta\cos(\bar{k}\bar{r})}{\bar{k}\bar{k}_1}\mathcal{F}_4^{II}(t)
$$

$$+ \frac{-3\bar{k}^2 + (3\bar{k}^2 + 20\beta^2\pi G_6)\cos(\bar{k}\bar{r})^2}{48\pi G_6 \beta \bar{k}^2 \bar{k}_1} \mathcal{F}_2^{II}(t)$$

$$+ \frac{3(6\beta^2\pi G_6\bar{k}r\sin(\bar{k}\bar{r}) - \cos(\bar{k}\bar{r})(\bar{k}^2 - 9\beta^2\pi G_6))}{8\bar{k}^3\bar{k}_1\beta\pi G_6} \left[\frac{\ddot{a}_0}{a_0} + \left(\frac{\dot{a}_0}{a_0}\right)^2\right]$$

$$- \frac{8\beta^2\pi G_6\bar{k}r\sin(\bar{k}\bar{r}) - \cos(\bar{k}\bar{r})(\bar{k}^2 - 12\beta^2\pi G_6)}{\bar{k}^3\bar{k}_1\beta}\delta\Lambda_6 \tag{A26}$$

$$A_\theta^{(1)\,III} = \mathcal{F}_7^{III}(t) - \frac{\beta\sin(\bar{k}\bar{r})}{\bar{k}_1}\mathcal{F}_6^{III}(t) + \frac{\beta\cos(\bar{k}\bar{r})}{\bar{k}\bar{k}_1}\mathcal{F}_4^{III}(t)$$

$$+ \frac{-3\bar{k}^2 + (3\bar{k}^2 + 20\beta^2\pi G_6)\cos(\bar{k}\bar{r})^2}{48\pi G_6 \beta \bar{k}^2 \bar{k}_1} \mathcal{F}_2^{III}(t)$$

$$+ \frac{3(6\beta^2\pi G_6\bar{k}r\sin(\bar{k}\bar{r}) - \cos(\bar{k}\bar{r})(\bar{k}^2 - 9\beta^2\pi G_6))}{8\bar{k}^3\bar{k}_1\beta\pi G_6} \left[\frac{\ddot{a}_0}{a_0} + \left(\frac{\dot{a}_0}{a_0}\right)^2\right]$$

$$- \frac{8\beta^2\pi G_6\bar{k}r\sin(\bar{k}\bar{r}) - \cos(\bar{k}\bar{r})(\bar{k}^2 - 12\beta^2\pi G_6)}{\bar{k}^3\bar{k}_1\beta}\delta\Lambda_6 \tag{A27}$$

$$A_\theta^{(1)\,IV} = \mathcal{F}_7^{IV}(t) - \frac{\beta\sin(k\hat{r})}{k}\mathcal{F}_6^{IV}(t) + \frac{\beta\cos(k\hat{r})}{k^2}\mathcal{F}_4^{IV}(t)$$

$$- \frac{(3k^2(k^2 - 8\sigma\pi G_6)\tan(k\hat{r})^2 - 4\beta^2\pi G_6(5k^2 - 16\sigma\pi G_6))\cos(k\hat{r})^2}{48\pi G_6 \beta k^5}\mathcal{F}_2^{IV}(t)$$

$$+ \frac{(4\beta^2\pi G_6 k r\sin(k\hat{r}) - \cos(k\hat{r})(k^2 - 6\beta^2\pi G_6))}{k^4\beta}\mathcal{F}_3^{IV}(t)$$

$$+ \frac{1}{48k^8\beta}\left[(-8k^2(k^2 - 8\sigma\pi G_6) - 4\beta^2\pi G_6(11k^2 - 16\sigma\pi G_6))\cos(k\hat{r})^3\right.$$

$$- (12k^2(k^2 + 8\sigma\pi G_6) - 3\beta^2\pi G_6(29k^2 + 80\sigma\pi G_6)))\cos(k\hat{r})$$

$$\left. - 12\pi G_6\beta^2 k(r(-3k^2 - 16\sigma\pi G_6) + \phi(-11k^2 + 16\sigma\pi G_6))\sin(k\hat{r})\right]\mathcal{P}_{*6}(t)$$

$$+ \frac{3(6\beta^2\pi G_6 k r\sin(k\hat{r}) - \cos(k\hat{r})(k^2 - 9\beta^2\pi G_6))}{8k^4\beta\pi G_6}\left[\frac{\ddot{a}_0}{a_0} + \left(\frac{\dot{a}_0}{a_0}\right)^2\right]$$

$$- \frac{\beta\pi G_6(2kr\sin(k\hat{r}) + 3\cos(k\hat{r}))}{2k^4}(\rho_* - 3p_*)$$

$$- \frac{8\beta^2\pi G_6 k r\sin(k\hat{r}) - \cos(k\hat{r})(k^2 - 12\beta^2\pi G_6)}{k^4\beta}\delta\Lambda_6 \tag{A28}$$

The conditions we will impose on the solutions are the following: they must all be regular at the poles, and must satisfy the jump conditions at the boundaries between the cores and the bulk. Furthermore, all solutions must match smoothly across the equator separating the upper and lower core, with the exception of the gauge field perturbations, which as explained in the main text, must be related by a single valued gauge transformation.

The important point to watch out for comes from the equation of motion for the gauge invariant variable $X(r,t)$, eq.(53). Indeed, this equation includes one dimensional delta functions coming from expanding the step function in terms of $\Delta r_0(t)$ and $\Delta r_*(t)$. Concretely, this tells us that the radial derivative of C_1 is not smooth across the boundaries, but rather obeys

$$\lim_{\epsilon\to 0}\left[C_1^{II'}(r_0 + \epsilon, t) - C_1^{I'}(r_0 - \epsilon, t)\right] = -8\pi G_6\sigma\Delta r_0(t) \tag{A29}$$

$$\lim_{\epsilon\to 0}\left[C_1^{IV'}(r_* + \epsilon, t) - C_1^{III'}(r_* - \epsilon, t)\right] = 8\pi G_6\sigma\Delta r_*(t) \tag{A30}$$

which we will regard as solving for $\Delta r_0(t)$ and $\Delta r_*(t)$ once all other functions have been set by the other boundary conditions. It is interesting to note that the appearance of one dimensional delta functions in our solutions is in line with the arguments recently presented in [56].

Once all boundary conditions have been imposed, we have explicit expressions for all the integration "constants" $\mathcal{F}_i(t)$ and we also have three additional equations which turn out to give one of the Friedmann equations (involving the combination $\ddot{a}_0/a_0 - (\dot{a}_0/a_0)^2$) and two conservation of energy equations (one for the energy density on each brane). Consistency between these three expressions allows us to recover the expression for the Hubble rate $(\dot{a}_0/a_0)^2$ in terms

of the energy densities. The expression written in terms of the six dimensional quantities are not very instructive however, so we will not write them out.

Armed with full solutions, we can now work out the effective four dimensional quantities, which as explained in the main text, are given by

$$\sigma^{(4)} = 2\pi \int_0^{r_0} e^{C_0(r)}\sigma dr = 2\pi \int_{\pi/k-\phi-r_0}^{\pi/k-\phi} e^{C_0(r)}\sigma dr \tag{A31}$$

$$\rho^{(4)}(t) = 2\pi \int_0^{r_0} e^{C_0(r)}\rho(t)dr + 2\pi \int_0^{r_0} e^{C_0(r)}\sigma C_1(r,t)dr + 2\pi\sigma e^{C_0(r_0)}\Delta r_0(t) \tag{A32}$$

$$p^{(4)}(t) = 2\pi \int_0^{r_0} e^{C_0(r)}p(t)dr - 2\pi \int_0^{r_0} e^{C_0(r)}\sigma C_1(r,t)dr - 2\pi\sigma e^{C_0(r_0)}\Delta r_0(t) \tag{A33}$$

$$\begin{aligned}
\rho_*^{(4)}(t) = {} & 2\pi \int_{\pi/k-\phi-r_0}^{\pi/k-\phi} e^{C_0(r)}\rho_*(t)dr + 2\pi \int_{\pi/k-\phi-r_0}^{\pi/k-\phi} e^{C_0(r)}\sigma C_1(r,t)dr \\
& - 2\pi\sigma e^{C_0(r_*)}\Delta r_*(t)
\end{aligned} \tag{A34}$$

$$\begin{aligned}
p_*^{(4)}(t) = {} & 2\pi \int_{\pi/k-\phi-r_0}^{\pi/k-\phi} e^{C_0(r)}p_*(t)dr - 2\pi \int_{\pi/k-\phi-r_0}^{\pi/k-\phi} e^{C_0(r)}\sigma C_1(r,t)dr \\
& + 2\pi\sigma e^{C_0(r_*)}\Delta r_*(t).
\end{aligned} \tag{A35}$$

Plugging these into the expressions for the Friedmann equations and conservation of energy allows us to derive eqs.(67-70).

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[57] Note that we will be using MTW [51] conventions throughout the paper.

[58] As will be seen in a subsequent section, consistency of the perturbative ansatz with coordinate invariance requires that some perturbations be $\mathcal{O}(\rho^{3/2})$ rather than $\mathcal{O}(\rho)$.

[59] In the bulk, we cannot exclude the possibility of warping, *i.e.,* nontrivial $N_0(r)$ and $A_0(r)$. However, we will content ourselves with perturbing around the unwarped solutions which, in contrast to the warped ones, can be put into closed form.

[60] We omit writing the $*$ quantities of the lower core, which transform in the same way.

[61] Counting degrees of freedom, the 11 can be understood as 5 from the metric, 6 from the stress energy tensor, plus $F_{t\theta}$ and $F_{r\theta}$, minus 2 from redefining r and t.

[62] Here, the equations we have written correspond to the following combinations of the Einstein and gauge field equations of motion: 49 is $(tt) - (xx)$, 50 is $(rr) - (\theta\theta)$, 51 is $F^{\theta b}_{;b}$, 52 is the (rr) Einstein equation, 53 is $-1/4(tt) - 3/4(xx) - 3/4(rr) + 3/4(\theta\theta)$, 54 is a combination of $S^a_{r;a}$ and $F^{\theta b}_{;b}$, 55 comes from $F_{[ab;c]} = 0$, 56 is $(tr) + (rt)$, 57 is (rt) and 58 is a combination of $S^a_{t;a}$ and $F^{\theta b}_{;b}$.

[63] The variables $\tilde{p}_6$ and $\tilde{p}_{*6}$ don't have a dynamical equation, so we are free to specify them arbitrarily in the absence of a specific model for the brane matter content.

[64] Strictly speaking, we should also define an effective 4D scale factor $a^{(4)} \sim \int dr\, b(r,t)\, c(r,t)\, a^{(6)}$. In the present case this is not necessary because there are no corrections to $\dot{a}/a$ and $\ddot{a}/a$ at leading order in the perturbation.

[65] The quantization condition then constrains the values the $U(1)$ coupling can take.

[66] These dominant terms in the thin brane limit come from the gauge invariant variable Z.